

RENORMALIZATION OF ALMOST COMMUTING PAIRS

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ABSTRACT. In this paper we give a new prove of hyperbolicity of renormalization of critical circle maps using the formalism of almost-commuting pairs. We extend renormalization to two-dimensional dissipative maps of the annulus which are small perturbations of one-dimensional critical circle maps. Finally, we demontsrate that a two-dimensional map which lies in the stable set of the renormalization operator possesses attractor which is topologically a circle. Such a circle is *critical*: the dynamics on it is topologically, but not smoothly, conjugate to a rigid rotation.

1. PRELIMINARIES

1.1. Introduction. Our motivation in this paper comes from the study of attractors of small two-dimensional perturbations of critical circle maps. Let us recall, that a critical circle map f is a C^3 -smooth orientation preserving homeomorphism of the circle $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$ which has a single critical point $x_0 \in \mathbb{T}$ whose order n is an odd integer. To fix the ideas, we will set $x_0 = 0$, and will assume that $n = 3$. By way of example, consider the two-parameter *Arnold's family*

$$f_{a,\omega}(x) = x - \frac{a}{2\pi} \sin 2\pi x + \omega.$$

Note that each $f_{a,\omega}$ commutes with the unit translation,

$$f_{a,\omega}(x + 1) = f_{a,\omega}(x) + 1,$$

and hence it projects to a well-defined map of the circle $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$, which we denote $\hat{f}_{a,\omega}$. For $|a| < 1$, this map is an analytic diffeomorphism, and for $|a| = 1$ it is a critical circle map. This illustrates the fact that a generic analytic homeomorphism of the circle which lies on the boundary of the set of analytic diffeomorphisms is a critical circle map.

For a circle homeomorphism f , we will denote $\rho(f) \in \mathbb{T}$ its rotation number. For a lift $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$, we obtain a representative of $\rho(f)$ given by $\lim \tilde{f}^n(x)/n$. We denote it by $\rho(\tilde{f}) \in \mathbb{R}$. As was shown by Yoccoz in [Yoc], every critical circle map f with $\rho(f) \notin \mathbb{Q}$ is topologically conjugate to the rigid rotation

$$R_{\rho(f)}(x) \equiv x + \rho(f) \bmod \mathbb{Z}.$$

Date: June 28, 2016.

Identifying $\rho(f)$ with its representative in $[0, 1)$, we can represent $\rho(f)$ as a continued fraction with positive terms

$$\rho(f) = \frac{1}{r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \dots}}} \quad (1.1)$$

For convenience, further on we will abbreviate this expression as $[r_0, r_1, r_2, \dots]$. The numbers r_i are determined uniquely if and only if $\rho(f)$ is irrational. In this case we shall say that $\rho(f)$ (or f itself) is of the type bounded by B if $\sup r_i \leq B$; it is of a periodic type if the sequence $\{r_i\}$ is periodic.

Let \mathbb{A}_r denote the annulus $\{| \operatorname{Im} z | < r\} / \mathbb{Z} \supset \mathbb{T}$ and let F be a real-analytic map

$$F : \mathbb{A}_r \rightarrow \mathbb{A}_r.$$

We let

$$\Lambda(F) = \bigcap_{n \in \mathbb{N}} F^n(\mathbb{A}_r),$$

and refer to it as the *attractor* of F ; we further call it a minimal attractor when the restriction $F|_\Lambda$ is minimal. In the case when f is a map of the circle, we can trivially extend it to the second coordinate, setting $F_f(x, y) = (f(x), 0)$; in this case, $\Lambda = \mathbb{T}$. Suppose, f is an analytic diffeomorphism of \mathbb{T} . Considerations of normal hyperbolicity imply that if G is a sufficiently small smooth perturbations of F_f , the attractor $\Lambda(G)$ is a smooth circle, and furthermore, when $\Lambda(G)$ is minimal, the dynamics of G on Λ is smoothly conjugate to the irrational rotation. Recently, E. Pujals [Puj] asked a question, whether, when considering small perturbations of critical circle maps, one would observe “critical” invariant circles: that is, topological circles $\Lambda(G)$ on which the dynamics is topologically, but not smoothly, conjugate to an irrational rotation. This question can be asked in a typical low-parameter family of perturbations of critical circle maps, or for a specific family of examples. Pujals proposed looking at the perturbed Arnold family, consisting of quotients under $x \equiv x + 1$ of maps of the form

$$(f_{a,\omega}(x) + y, \epsilon(f_{a,\omega} - x + y)),$$

where ϵ is a small parameter. Here, if we, for instance, fix the rotation number

$$\rho_* = (\sqrt{5} - 1)/2 = [1, 1, 1, 1, \dots],$$

one would expect that possessing a critical circle with rotation number ρ_* would be a codimension 2 phenomenon, occurring on the boundary of the set in which the Λ is a non-critical circle with the same rotation number.

In this paper, we confirm that critical circles exist in typical families, and explain the criticality phenomenon in terms of hyperbolicity of renormalization, which is a subject of this paper in its own right. Briefly, maps of the annulus with a critical circle with rotation number ρ_* (for example) lie in the stable manifold of the one-dimensional hyperbolic fixed point of renormalization.

Of course, renormalization of critical circle maps is a classical subject, and one of the central themes in the development of modern one-dimensional dynamics. We refer the reader to the papers [Ya3, Ya4] of the second author in which the main renormalization conjectures, known as Lanford’s Program, were proved. The preceding historical development of the subject is described in [Ya3]. The “classical” definition of renormalization of critical circle maps uses the language of *commuting pairs*, as described below. Analytic commuting pairs provided the setting for proving the existence of renormalization horseshoe attractor [dF2, dFdM2, Ya4]. However, there was a conceptual difficulty in proving hyperbolicity in this setting, as the space of analytic commuting pairs does not possess a natural structure of a Banach manifold.

This difficulty was finessed by the second author using a concept of *cylinder renormalization*, introduced in [Ya3]. Cylinder renormalization operator \mathcal{R}_{cyl} has two key properties, necessary for the study of hyperbolic properties of the renormalization horseshoe attractor:

- (1) \mathcal{R}_{cyl} acts on a Banach manifold (of analytic maps of the circle, whose domain of analyticity includes a certain fixed annulus);
- (2) the operator \mathcal{R}_{cyl} is smooth (in fact, analytic).

Cylinder renormalization has since become an important tool in one-dimensional renormalization theory. It applies to analytic maps with Siegel disks [Ya5, GaY]; in the limiting case it becomes the all-important *parabolic renormalization* [EY, IS]; and very recently it has been applied to the study of critical circle maps with non-integer critical exponents [GoY].

However, the question of proving hyperbolicity in the setting of commuting pairs has remained relevant. One of the main reasons for this is that cylinder renormalization does not extend readily to small two-dimensional perturbations of critical circle maps. The definition of \mathcal{R}_{cyl} relies on the Uniformization Theorem of doubly-connected domains of one-dimensional Complex Analysis. This definition does not naturally generalize to two-dimensional maps. In this paper, we revisit the problem of hyperbolicity of renormalization. As will be seen in the next section, we use a “classical” definition of renormalization and the definition of a Banach manifold in which renormalization becomes smooth (analytic) – and thus satisfy the above conditions (1)-(2) for commuting pairs.

We then give a new proof of renormalization hyperbolicity – in the “classical” setting of commuting pairs. This allows us to apply our renormalization to small two-dimensional perturbations of critical circle maps. We find a suitable smooth extension of renormalization to dissipative maps of the annulus in two dimension, and prove renormalization hyperbolicity for such maps. Finally, we apply our renormalization results to the study of dissipative attractors of small perturbations of critical circle maps, to prove a version of Pujals’ conjectures.

1.2. Commuting pairs. As discussed in some detail in [Ya3], the space of critical circle maps is ill-suited to define renormalization. The pioneering works on the subject ([ORSS] and [FKS]) circumvented this difficulty by replacing critical circle maps with different objects:

Definition 1.1. A C^r -smooth (or C^ω) critical commuting pair $\zeta = (\eta, \xi)$ consists of two C^r -smooth (or C^ω) orientation preserving interval homeomorphisms $\eta : I_\eta \rightarrow \eta(I_\eta)$, $\xi : I_\xi \rightarrow \xi(I_\xi)$, where

- (I) $I_\eta = [0, \xi(0)]$, $I_\xi = [\eta(0), 0]$;
- (II) Both η and ξ have homeomorphic extensions to interval neighborhoods of their respective domains *with the same degree of smoothness*, that is C^r (or C^ω), which commute, $\eta \circ \xi = \xi \circ \eta$;
- (III) $\xi \circ \eta(0) \in I_\eta$;
- (IV) $\eta'(x) \neq 0 \neq \xi'(y)$, for all $x \in I_\eta \setminus \{0\}$, and all $y \in I_\xi \setminus \{0\}$;
- (V) each of the maps η and ξ has a cubic critical point at 0:

$$\eta'(0) = \eta''(0) = \xi'(0) = \xi''(0) = 0, \text{ and } \eta'''(0) \neq 0 \neq \xi'''(0).$$

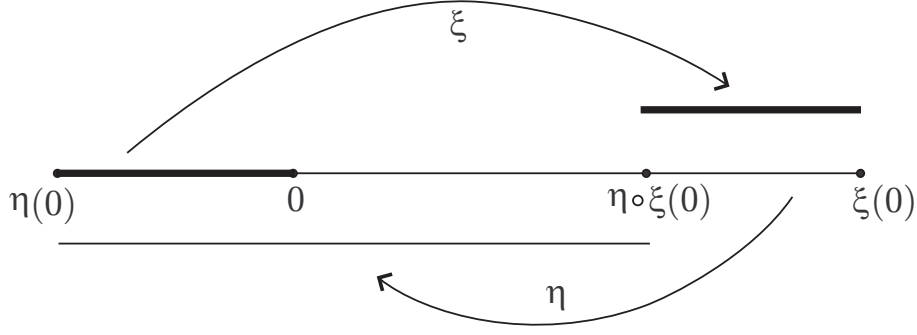


FIGURE 1. A commuting pair

The commutation condition allows one to “seamlessly” iterate the extensions of the maps of a commuting pair.

Given a critical commuting pair $\zeta = (\eta, \xi)$ we can regard the interval $I = [\eta(0), \xi \circ \eta(0)]$ as a circle, identifying $\eta(0)$ and $\xi \circ \eta(0)$ and define $f_\zeta : I \rightarrow I$ by

$$f_\zeta = \begin{cases} \eta \circ \xi(x) & \text{for } x \in [\eta(0), 0] \\ \eta(x) & \text{for } x \in [0, \xi \circ \eta(0)] \end{cases}$$

The mapping ξ extends to a C^r - (or C^ω -) diffeomorphism of open neighborhoods of $\eta(0)$ and $\xi \circ \eta(0)$. Using it as a local chart we turn the interval I into a closed one-dimensional manifold M . Condition (II) above implies that the mapping f_ζ projects to a well-defined C^3 -smooth homeomorphism $F_\zeta : M \rightarrow M$. Identifying M with the circle by a diffeomorphism $\phi : M \rightarrow \mathbb{T}$ we recover a critical circle mapping $f^\phi = \phi \circ F_\zeta \circ \phi^{-1}$. The critical circle mappings corresponding to two different choices of ϕ are conjugated by a diffeomorphism, and thus we recovered a C^r - (or C^ω) smooth conjugacy class of circle mappings from a critical commuting pair.

Let f be a critical circle mapping, whose rotation number ρ has a continued fraction expansion (1.1) with at least $m + 1$ terms, and let $p_m/q_m = [r_0, \dots, r_{m-1}]$. The pair of iterates $f^{q_{m+1}}$ and f^{q_m} restricted to the circle arcs I_m and I_{m+1} correspondingly can be viewed as a critical commuting pair in the following way. Let \bar{f} be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m . Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ denote the translation $x \mapsto x + 1$. Let $\eta : \bar{I}_m \rightarrow \mathbb{R}$, $\xi : \bar{I}_{m+1} \rightarrow \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|_{\bar{I}_m}, \xi|_{\bar{I}_{m+1}})$ forms a critical commuting pair corresponding to $(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}})$. Henceforth we shall simply denote this commuting pair by

$$(f^{q_{m+1}}|_{I_m}, f^{q_m}|_{I_{m+1}}). \quad (1.2)$$

The *height* $\chi(\zeta)$ of a critical commuting pair $\zeta = (\eta, \xi)$ is equal to r , if

$$0 \in [\eta^r(\xi(0)), \eta^{r+1}(\xi(0))].$$

If no such r exists, we set $\chi(\zeta) = \infty$, in this case the map $\eta|_{I_\eta}$ has a fixed point. For a pair ζ with $\chi(\zeta) = r < \infty$ one verifies directly that the mappings $\eta|_{[0, \eta^r(\xi(0))]}$ and $\eta^r \circ \xi|_{I_\xi}$ again form a commuting pair. For a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\tilde{\zeta}$ the pair $(\tilde{\eta}|_{\tilde{I}_\eta}, \tilde{\xi}|_{\tilde{I}_\xi})$ where tilde means rescaling by the linear factor $\lambda = -1|I_\eta|$:

$$\tilde{\zeta}(z) = \lambda^{-1}\zeta(\lambda z).$$

Definition 1.2. We say that a real commuting pair $\zeta = (\eta, \xi)$ is *renormalizable* if $\chi(\zeta) < \infty$. The *renormalization* of a renormalizable commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi|_{I_\xi}}, \widetilde{\eta|_{[0, \eta^r(\xi(0))]}]).$$

The non-rescaled pair $(\eta^r \circ \xi|_{I_\xi}, \eta|_{[0, \eta^r(\xi(0))]}])$ will be referred to as the *pre-renormalization* $p\mathcal{R}\zeta$ of the commuting pair $\zeta = (\eta, \xi)$. Suppose $\{\zeta_i\}_{i=1}^{k-1}$ is a sequence of renormalizable pairs such that $\zeta_0 = \zeta$ and $\zeta_i = p\mathcal{R}\zeta_{i-1}$. We call $\zeta_k = p\mathcal{R}\zeta_{k-1}$ the k -th pre-renormalization of ζ ; and $\tilde{\zeta}_k$ the k -th renormalization of ζ and write

$$\zeta_k = p\mathcal{R}^k\zeta, \quad \tilde{\zeta}_k = \mathcal{R}^k\zeta.$$

Let $\zeta_k = (\eta_k, \xi_k)$. The domains of η_k and ξ_k will be denoted I_k and J_k correspondingly.

For a pair ζ we define its *rotation number* $\rho(\zeta) \in [0, 1]$ to be equal to the continued fraction $[r_0, r_1, \dots]$ where $r_i = \chi(\mathcal{R}^i\zeta)$. In this definition $1/\infty$ is understood as 0, hence a rotation number is rational if and only if only finitely many renormalizations of ζ are defined; if $\chi(\zeta) = \infty$, $\rho(\zeta) = 0$. Thus defined, the rotation number of a commuting pair can be viewed as a rotation number in the usual sense:

Proposition 1.1. *The rotation number of the mapping F_ζ is equal to $\rho(\zeta)$.*

There is an advantage in defining $\rho(\zeta)$ using a sequence of heights in removing the ambiguity in prescribing a continued fraction expansion to rational rotation numbers in a renormalization-natural way.

1.3. Dynamical partitions and real *a priori* bounds. We need to recall the definition of a dynamical partition, which becomes somewhat technical in the language of commuting pairs. Consider the space \mathcal{I} of multi-indices $\bar{s} = (a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ where $a_j \in \mathbb{N}$ for $2 \leq n$, $a_1 \in \mathbb{N} \cup \{0\}$, $b_j \in \mathbb{N}$ for $1 \leq j \leq n-1$, and $b_n \in \mathbb{N} \cup \{0\}$. We introduce a partial ordering on multi-indices: $\bar{s} \succ \bar{t}$ if $\bar{s} = (a_1, b_1, a_2, b_2, \dots, a_n, b_n)$, $\bar{t} = (a_1, b_1, \dots, a_k, b_k, c, d)$, where $k < n$ and either $c < a_{k+1}$ and $d = 0$ or $c = a_{k+1}$ and $d < b_{k+1}$. For such a pair, we also define

$$\bar{q} \equiv \bar{s} \ominus \bar{t} :$$

- in the case when $d = 0$, $\bar{q} = (a_{k+1} - c, b_{k+1}, \dots, a_n, b_n)$;
- in the other case, $\bar{q} = (0, b_{k+1} - d, a_{k+1}, b_{k+2}, \dots, a_n, b_n)$.

Let us define the n -th dynamical partition \mathcal{P}_n of $\zeta = (\eta, \xi)$ which is at least n times renormalizable. Namely, consider the n -th pre-renormalization

$$\zeta_n = (\eta_n|_{I_n}, \xi_n|_{J_n}), \text{ where } I_n = [0, \xi_n(0)] \text{ and } J_n = [0, \eta_n(0)].$$

Here

$$\eta_n = \zeta^{\bar{s}_n}, \text{ and } \xi_n = \zeta^{\bar{t}_n} \text{ for some } \bar{s}_n, \bar{t}_n \in \mathcal{I}.$$

Now consider the collection of intervals

$$\mathcal{P}_n \equiv \{\zeta^{\bar{w}}(I_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } \zeta^{\bar{w}}(J_n) \text{ for all } \bar{w} \prec \bar{t}_n\}.$$

It is easy to see that:

- (a) $\bigcup_{H \in \mathcal{P}_n} H = [\eta(0), \xi(0)]$;
- (b) for any two distinct elements H_1 and H_2 of \mathcal{P}_n , the interiors of H_1 and H_2 are disjoint.

We denote $\overline{\mathcal{P}_n}$ the set of boundary points of the n -th dynamical partition.

For a pair of maps $\zeta = (\eta, \xi)$ and \bar{s} as above we will denote

$$\zeta^{\bar{s}} \equiv \xi^{b_n} \circ \eta^{a_n} \circ \dots \circ \xi^{b_2} \circ \eta^{a_2} \circ \xi^{b_1} \circ \eta^{a_1}.$$

Similarly,

$$\zeta^{-\bar{s}} \equiv (\zeta^{\bar{s}})^{-1} = (\eta^{a_1})^{-1} \circ (\xi^{b_1})^{-1} \circ \dots \circ (\eta^{a_n})^{-1} \circ (\xi^{b_n})^{-1}.$$

Successive renormalizations of a C^3 -smooth commuting pair with an irrational rotation number form a pre-compact family, all of the limit points of which are *analytic*. For a strong version of this statement, known as *real a priori bounds*, see [dFdM1]; we will need the following consequence of compactness:

Proposition 1.2. *There exists a universal constant $C_0 > 1$ such that the following holds. Let S be a compact set of C^3 -smooth commuting pairs (note that S could consist of a single pair). Then there exists $N = N(S)$ such that for all $n \geq N$ the following holds. Let $\zeta \in S$ be at least n times renormalizable. Let I and J be two adjacent intervals of the n -th dynamical partition of ζ . Then I and J are C_0 -commensurable:*

$$\frac{1}{C_0}|I| < |J| < C_0|I|.$$

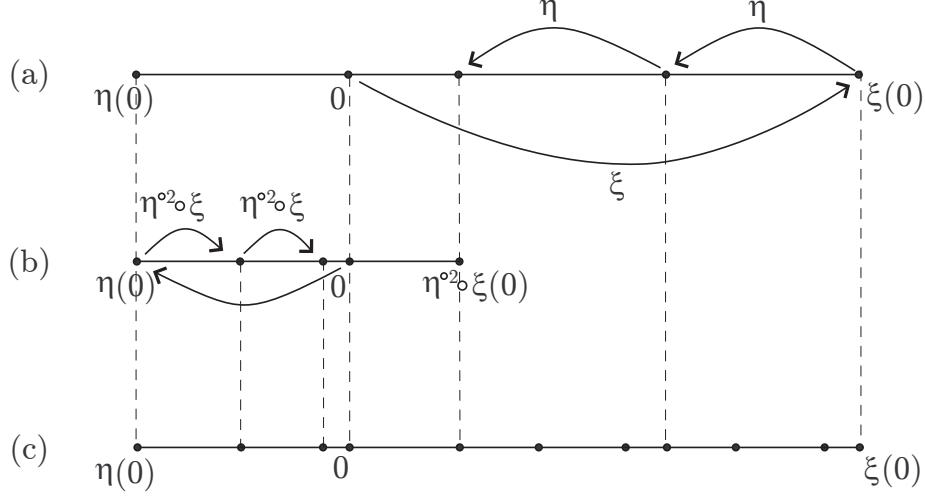


FIGURE 2. The 1-st and 2-nd dynamical partitions for a pair ζ with $\rho(\zeta) = [2, 2, \dots]$: (a) forming the partition of level 1; (b) the 1-st dynamical partition of the pre-renormalization $p\mathcal{R}\zeta$; (c) the 2-nd dynamical partition of ζ .

In particular, denoting $p\mathcal{R}^n\zeta = (\eta', \xi')$, we have

$$\frac{1}{C_0}|I_{\xi'}| < |I_{\eta'}| < C_0|I_{\xi'}|.$$

1.4. Renormalization horseshoe. In [Ya2] we constructed a horseshoe attractor for renormalization of analytic maps. Denote $\bar{\Sigma}$ the space of bi-infinite sequences

$$(\dots, r_{-k}, \dots, r_{-1}, r_0, r_1, \dots, r_k, \dots) \text{ with } r_i \in \mathbb{N} \cup \{\infty\}$$

equipped with the weak topology.

Theorem 1.3 (Renormalization horseshoe). *There exists an \mathcal{R} -invariant set \mathcal{X} consisting of C^ω -smooth commuting pairs with irrational rotation numbers with the following properties. The operator \mathcal{R} continuously extends to the closure*

$$\mathcal{A} \equiv \overline{\mathcal{X}}$$

and the action of \mathcal{R} on \mathcal{A} is topologically conjugate to the two-sided shift $\sigma : \bar{\Sigma} \rightarrow \bar{\Sigma}$:

$$i \circ \mathcal{R} \circ i^{-1} = \sigma$$

so that if $\zeta = i^{-1}(\dots, r_{-k}, \dots, r_{-1}, r_0, r_1, \dots, r_k, \dots)$ then $\rho(\zeta) = [r_0, r_1, \dots, r_k, \dots]$. For any analytic commuting pair ζ with an irrational rotation number we have

$$\mathcal{R}^n\zeta \rightarrow \mathcal{A}$$

uniformly on compact sets. Moreover, for any two analytic commuting pairs ζ, ζ' with $\rho(\zeta) = \rho(\zeta')$ we have

$$\text{dist}(\mathcal{R}^n \zeta, \mathcal{R}^n \zeta') \rightarrow 0$$

for the uniform distance between analytic extensions of the renormalized pairs on compact sets.

We will denote \mathcal{A}_B the subset of the attractor consisting of pairs with rotation numbers of a type bounded by B . Its existence, and the corresponding version of Theorem 1.3 was shown by E. de Faria (see [dF1, dF2] and also [dFdM2]).

Let $\zeta = (\eta, \xi)$ be a commuting pair such that $\xi(0) = 1$. Denote $C^0([0, 1])$ the Banach space of bounded C^0 functions on the interval $[0, 1]$ with the uniform norm. We can identify ζ with a point in $\mathbb{R} \times C^0([0, 1]) \times C^0([0, 1])$ by

$$\zeta \mapsto (\eta(0), \eta(x), \frac{1}{\eta(0)} \xi(\eta(0)x)). \quad (1.3)$$

This induces a distance on the set of commuting pairs, which we denote dist_{C^0} . We note that the following has been recently proven by W. de Melo and P. Guarino [dMG]:

Theorem 1.4. *There exists $\delta > 0$ such that the following holds. Let ζ_1 and ζ_2 be two C^3 -smooth commuting pairs with the same irrational rotation number $\rho = \rho(\zeta_1) = \rho(\zeta_2)$ of bounded type. Then there exists $C > 0$ such that*

$$\text{dist}_{C^0}(\mathcal{R}^n \zeta_1, \mathcal{R}^n \zeta_2) < C(1 + \delta)^{-n}.$$

1.5. Spaces of analytic almost commuting pairs. Because of the commutation condition, there is no natural Banach manifold structure on the space of analytic commuting pairs.

However, there is one on the space of C^r -smooth commuting pairs with $r \geq 3$, considered modulo an affine conjugacy. Indeed, pick the unique representative $\zeta = (\eta, \xi)$ of an affine conjugacy class, which is given by the normalization $\xi(0) = 1$. Let $C^r([0, 1])$ denote the Banach space of C^r -smooth functions on $[0, 1]$ with the norm

$$\|f\|_{C^r} = \sum_{k=0}^r \sup_{x \in [0, 1]} \left| \frac{d^k}{dx^k} f \right|.$$

As above, identify C^r -smooth commuting pairs with a subset of $\mathbb{R} \times C^r([0, 1]) \times C^r([0, 1])$ via (1.3). It is possible to show that this subset has a submanifold structure. Clearly, the space of C^r -smooth commuting pairs is renormalization-invariant. However, it is an elementary exercise to show that the operator \mathcal{R} is *not* differentiable in the space of C^r -smooth pairs (indeed, composition, considered as an operator $C^r \times C^r \rightarrow C^r$ is not differentiable). Thus the setting of C^r -smooth commuting pairs is equally unsuitable for the study of the hyperbolic properties of \mathcal{R} .

We, therefore, take a different path. The principal object in our approach to critical circle maps is the following space:

Definition 1.3. The space \mathbf{B} consists of C^3 -smooth commuting pairs $\zeta = (\eta, \xi)$, such that the maps η, ξ are complex-analytic on some neighborhoods of their intervals of definition.

We call the elements of \mathbf{B} *analytic almost commuting pairs* or simply *almost commuting pairs*. A version of this “classical” approach was first used in the computer-assisted proof of renormalization hyperbolicity by Mestel [Mes], although, it has not received any further development in the literature since.

We claim that an equivalent way of describing this space is the following:

Definition 1.4. The space \mathbf{B} consists of pairs of non-decreasing interval maps

$$\eta : [0, \xi(0)] \rightarrow [\eta(0), \eta \circ \xi(0)], \quad \xi : [\eta(0), 0] \rightarrow [\xi \circ \eta(0), \xi(0)]$$

which have the following properties:

- (1) there exists an open neighborhood of the interval $[0, \xi(0)]$ on which the map η is analytic, with a single critical point of order 3 at the origin;
- (2) similarly, there exists an open neighborhood of the interval $[\eta(0), 0]$ on which the map ξ is analytic, with a single critical point of order 3 at the origin;
- (3) the commutator

$$[\eta, \xi](x) \equiv \eta \circ \xi(x) - \xi \circ \eta(x) = o(x^3) \text{ at } x = 0.$$

It is evident that a pair satisfying Definition 1.3 also satisfies Definition 1.4. To prove the converse, let (η, ξ) be a pair satisfying 1.4. Consider the extension of η to a function $\tilde{\eta}$ defined in a neighborhood of 0, which is given by η on $[0, \xi(0)]$ and by $\xi^{-1} \circ \eta \circ \xi$ on $[\eta(0), 0]$. Since ξ is a local diffeomorphism away from the origin, we have

$$\eta(x) - \tilde{\eta}(x) \sim \xi \circ \eta(x) - \eta \circ \xi(x) = o(x^3).$$

Hence, $\tilde{\eta}$ is a C^3 -smooth extension of η to a neighborhood of $[0, \xi(0)]$, which commutes with the analytic extension of ξ , and the claim is proved.

Suppose, B is a complex Banach space whose elements are functions of a complex variable. Let us say that the *real slice* of B is the real Banach space $B^{\mathbb{R}}$ consisting of the real-symmetric elements of B . If X is a Banach manifold modelled on B with the atlas $\{\Psi_\gamma\}$ we shall say that X is *real-symmetric* if $\Psi_{\gamma_1} \circ \Psi_{\gamma_2}^{-1}(B^{\mathbb{R}}) \subset B^{\mathbb{R}}$ for any pair of indices γ_1, γ_2 . The *real slice of X* is then defined as the real Banach manifold $X^{\mathbb{R}} \subset X$ given by $\Psi_\gamma^{-1}(B^{\mathbb{R}})$ in a local chart Ψ_γ . An operator A defined on a subset of X is *real-symmetric* if $A(X^{\mathbb{R}}) \subset X^{\mathbb{R}}$.

Definition 1.5. For a choice of topological disks $D \supset [0, 1]$, E , we let $\mathbf{B}_0^{D,E}$ consists of pairs in \mathbf{B} whose maps η and ξ have bounded analytic continuations to D and E correspondingly, such that $[\eta(0), 0] \subset E$. We view it as a subset of the real slice of the complex Banach space $C^\omega(D) \times C^\omega(E)$ where $C^\omega(W)$ denotes the space of bounded holomorphic functions on W with the uniform norm. Finally, denote $\mathbf{B}^{D,E}$ the space of pairs in $\mathbf{B}_0^{D,E}$ with further normalization conditions $\xi(0) = 1$, and $\frac{1}{2C_0} < |\eta(0)| < 2C_0$, where C_0 is as in Proposition 1.2.

Proposition 1.5. *With these norms, the space $\mathbf{B}^{D,E}$ is a real Banach manifold, modeled on a finite-codimensional subspace of the real slice of the Banach space $C^\omega(D) \times C^\omega(E)$.*

Proof. Firstly, note that the conditions $\eta'(0) = \eta''(0) = \xi'(0) = \xi''(0) = 0$ define a Banach subspace of $C^\omega(D) \times C^\omega(E)$. Furthermore, by the Argument Principle, the conditions $\eta'''(0) \neq 0$, $\xi'''(0) \neq 0$ and $\eta'(x) > 0$ and $\xi'(x) > 0$ on the real line in proper subneighborhoods of D and E respectively, define an open subset \mathcal{W} of this Banach subspace.

Now, consider the commutation conditions. The conditions $\eta'(0) = \eta''(0) = \xi'(0) = \xi''(0) = 0$ imply that

$$(\eta \circ \xi)^{(n)}(0) = (\xi \circ \eta)^{(n)}(0) = 0 \text{ for } n = 1, 2.$$

Let $k \geq 3$, and write

$$\eta(x) = \eta_k x^k + \eta_{k+1} x^{k+1} + q(x), \quad \xi(x) = \xi_0 + s(x), \quad \text{where } \xi_0 \equiv \xi(0), \quad \eta_k \equiv \eta^{(k)}(0)/k!.$$

Note that a tuple

$$(\eta_k, \eta_{k+1}, \xi_0, q(x), s(x))$$

forms a set of analytic coordinates in the real slice of $C^\omega(D) \times C^\omega(E)$. We will demonstrate that locally $\mathbf{B}^{D,E}$ is analytically parametrized by $(\eta_k, \eta_{k+1}, \xi_0)$ for some choice of k . To this end, consider the map

$$\mathbf{F} : \bar{\mathbf{B}}_0^{D,E} \mapsto \mathbb{C}^3$$

given by

$$\begin{aligned} F_1(\eta_k, \eta_{k+1}, \xi_0; q, s) &\equiv \eta(\xi(0)) - \xi(\eta(0)) = \eta_k \xi_0^k + \eta_{k+1} \xi_0^{k+1} + q(\xi_0) - \xi_0 - s(\eta_0), \\ F_2(\eta_k, \eta_{k+1}, \xi_0; q, s) &\equiv (\eta \circ \xi)'''(0) - (\xi \circ \eta)'''(0) = \eta'(\xi_0) \xi'''(0) - \xi'(\eta_0) \eta'''(0) = \\ &= (k \eta_k \xi_0^{k-1} + (k+1) \eta_{k+1} \xi_0^k + q'(\xi_0)) \xi'''(0) - \\ &\quad \xi'(\eta_0) (k(k-1)(k-2) \eta_k \xi_0^{k-3} + (k+1)(k-1) k \eta_{k+1} \xi_0^{k-2} + q'''(\xi_0)) \\ F_3(\eta_k, \eta_{k+1}, \xi_0; n, s) &\equiv \xi_0 - 1. \end{aligned}$$

Thus, $\mathbf{B}^{D,E} = \mathbf{F}^{-1}(0)$.

We have that

$$\begin{aligned} D_{\eta_k, \eta_{k+1}} \mathbf{F} &\equiv \begin{bmatrix} \frac{\partial F_1}{\partial \eta_k} & \frac{\partial F_1}{\partial \eta_{k+1}} \\ \frac{\partial F_2}{\partial \eta_k} & \frac{\partial F_2}{\partial \eta_{k+1}} \end{bmatrix} = \begin{bmatrix} \xi_0^k & \xi_0^{k+1} \\ k \xi_0^{k-1} \xi'''(0) - \xi'(\eta_0) k(k-1)(k-2) \xi_0^{k-3} & (k+1) \xi_0^k \xi'''(0) - \xi'(\eta_0) (k+1)(k-1) k \xi_0^{k-2} \end{bmatrix} \\ \implies \det(D_{\eta_k, \eta_{k+1}, \xi_0} \mathbf{F}(\eta_k, \eta_{k+1}, 1; q, s)) &= \xi'''(0) - 3k(k-1) \xi'(\eta_0). \end{aligned}$$

Let $\zeta_0 = (\eta_0, \xi_0) \in \mathbf{B}^{D,E}$. Then there exists a neighborhood $\mathcal{U}(\zeta_0) \subset \mathcal{W}$ in which $|\xi'''(0)|$ is bounded from above and $\xi'(\eta_0)$ is bounded away from zero. Hence, there exists $k \geq 3$ such that in $\mathcal{U}(\zeta_0)$ the above determinant is non-zero. By Regular Value Theorem this implies the desired result. \square

We will denote $\mathfrak{B}^{D,E}$ the complex Banach manifolds of pairs defined in the same way as $\mathbf{B}^{D,E}$, but without the condition of real symmetry, so that

$$\mathbf{B}^{D,E} = (\mathfrak{B}^{D,E})^{\mathbb{R}}.$$

Our first statement is:

Proposition 1.6. *The space \mathbf{B} are renormalization invariant: let $\zeta \in \mathbf{B}$ and $\rho(\zeta) \neq 0$. Then $\mathcal{R}(\zeta) \in \mathbf{B}$. Moreover, let $\rho(\zeta) \notin \mathbb{Q}$. Then*

$$\mathcal{R}^n(\zeta) \rightarrow \mathcal{A}$$

at a geometric rate, where \mathcal{A} is the hyperbolic horseshoe attractor of renormalization constructed in [Ya2, Ya4].

Proof. The space of C^3 -smooth commuting pairs is \mathcal{R} -invariant, and the geometric convergence statement holds on this space (see [dF1, dF2, dFdM1]). Preservation of the other properties of pairs in \mathbf{B} is evident from the definition of \mathcal{R} . \square

1.6. Complex *a priori* bounds.

Definition 1.6. For $0 < \mu < 1$ and $K > 1$ let us denote $\mathbf{H}(\mu, K)$ the set of almost commuting pairs with the following properties:

- there exist topological disks U , V and Δ which contain the origin and such that U and V are compactly contained in Δ and

$$\eta : U \rightarrow (\Delta \setminus \mathbb{R}) \cup \eta(U \cap \mathbb{R}) \text{ and } \xi : V \rightarrow (\Delta \setminus \mathbb{R}) \cup \xi(V \cap \mathbb{R})$$

are three-fold branched coverings;

- let A be the maximal annulus separating $\mathbb{C} \setminus \bar{\Delta}$ from $U \cup V$. Then $\text{mod } A > \mu$;
- $\xi(0) = 1$ and $\mu < \eta(0) < 1/\mu$;
- $[0, 1] \subset U$ and $[\eta(0), 0] \subset V$;
- $\text{diam}(\Delta) < 1/\mu$ and Δ is a K -quasidisk.

Lemma 1.7 (Lemma 2.15 [Ya2]). *For each $\mu > 0$ the space $\mathbf{H}(\mu, K)$ is sequentially pre-compact in the Carathéodory topology, with every limit point contained in $\mathbf{H}(\mu/2, 2K)$.*

Theorem 1.8. *There exists universal constants $\mu > 0$ and $K > 1$ such that the following holds. Let $S \subset \mathbf{B}$ be a compact subset. Then there exists $N = N(S)$ such that for every almost commuting pair $\zeta \in S$ which is $n \geq N$ times renormalizable, the renormalization*

$$\zeta_n = R^n \zeta \in \mathbf{H}(\mu, K).$$

Furthermore, there exists a universal $R > 1$ such that the range Δ_n of ζ_n can be chosen as $\Delta_n = D_R(0)$.

The proof of this theorem was first given by the second author in [Ya1] for C^ω -commuting pairs in the Epstein class, and was later adapted in [dFdM2] for C^ω -commuting pairs without the Epstein property. However, these arguments do not use commutativity of the

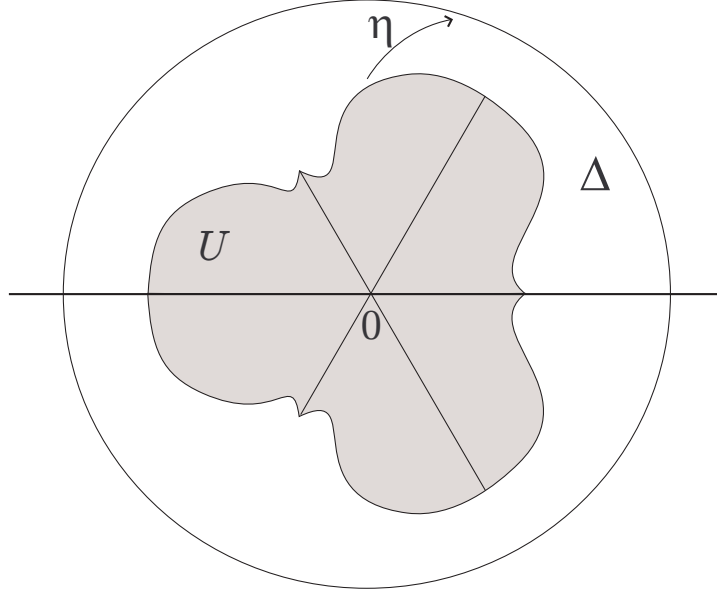


FIGURE 3. An extension of η as a 3-fold branched covering map $U \rightarrow \Delta$. The preimage of the real line is indicated.

pair beyond order zero (i.e. $\eta \circ \xi(0) = \xi \circ \eta(0)$). Hence, the theorem holds in the above generality.

We conclude this section with the following statement which is an immediate consequence of Theorem 1.8 and the compactness statement of Lemma 1.7:

Theorem 1.9. *There exists a space $\mathbf{B}^{D,E}$ and $m \in \mathbb{N}$ such that the following holds. Let $\zeta \in \mathbf{B}^{D,E}$ be an m -times renormalizable almost commuting pair. There exist larger domains $D' \supset D$, and $E' \supset E$ so that*

$$\mathcal{R}^m(\zeta) \in \mathbf{B}^{D',E'}.$$

Proof. Let $0 < \mu < 1$, $K > 1$ be as in Theorem 1.8. By Lemma 1.7, there exist $\mathbf{B}^{D',E'}$ such that $\mathbf{H}(\mu, K) \subset \mathbf{B}^{D',E'}$. Let us fix $\mathbf{B}^{D,E}$ so that $D' \supset D$, and $E' \supset E$. By Koebe Distortion Theorem, the set $\mathbf{B}^{D,E}$ is compact in \mathbf{B} in the C^3 -metric on the real line. This implies that the constant N in Theorem 1.7 can be chosen uniformly in $\mathbf{B}^{D,E}$. To complete the proof, let $m \geq N$. \square

2. HYPERBOLICITY OF RENORMALIZATION IN ONE DIMENSION

This section is devoted to proving the following theorem:

Theorem 2.1. *Let us fix a periodic point $\zeta_* \in \mathcal{A}$ of \mathcal{R} of period k and let $\rho_* = \rho(\zeta_*)$. There exists a space $\mathbf{B}^{D,E}$ and $p = m \cdot k \in \mathbb{N}$ such that the following holds. The pair ζ_* is*

a fixed point of \mathcal{R}^p in the space $\mathbf{B}^{D,E}$. The operator

$$\mathcal{R}^p(\zeta_*) \in \mathbf{B}^{D',E'} \text{ where } D' \ni D, E' \ni E.$$

The linearization

$$\mathcal{L} \equiv D\mathcal{R}^p|_{\zeta_*}$$

in $\mathbf{B}^{D,E}$ is a compact operator with one simple unstable eigenvalue, and the rest of the spectrum is compactly contained in \mathbb{D} . The stable manifold $\mathcal{W}^s(\zeta_*)$ of ζ_* contains all pairs in $\mathbf{B}^{D,E}$ with the rotation number ρ_* .

Let $\zeta \in \mathcal{W}^s(\zeta_*)$ and consider its n -th prerenormalization $\zeta_n = (\zeta^{\bar{s}_n}, \zeta^{\bar{t}_n})$ defined on linear rescalings D_n and E_n of the sets D and E correspondingly. Consider the collection of topological disks

$$\mathcal{V}_n \equiv \{\zeta^{\bar{w}}(D_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } \zeta^{\bar{w}}(E_n) \text{ for all } \bar{w} \prec \bar{t}_n\}.$$

We will refer to this collection of sets the n -th complex dynamical partition of ζ . It is clear from the construction that the elements $\zeta^{\bar{w}}(I_n)$ and $\zeta^{\bar{w}}(J_n)$ of the dynamical partition \mathcal{P}_n are contained in the elements $\zeta^{\bar{w}}(D_n)$ and $\zeta^{\bar{w}}(E_n)$, respectively, of the complex dynamical partition \mathcal{V}_n . Set $\lambda_n = (-1)^n |I_n|$ so that

$$\mathcal{R}^n \zeta(z) = \lambda_n^{-1} p \mathcal{R}^n \zeta(\lambda_n z).$$

As a consequence of Theorem 2.1 we have the following:

Corollary 2.2. *Let ζ_* be as in Theorem 2.1. Let $\zeta \in \mathcal{W}^s(\zeta_*)$. Then there exists $N = N(\zeta)$, $C > 0$, $C' > 0$, $K > 0$ and $0 < \gamma < 1$ so that for every $n > N$ the following holds.*

- 1) *If $Q_n \in \mathcal{V}_n$ then $\text{diam}(Q_n) < C\gamma^n$.*
- 2) *Any two neighboring domains $Q_n, Q'_n \in \mathcal{V}_n$ are K -commensurate.*
- 3) *For every $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) set $\psi_{\bar{w}}^\zeta = \zeta^{\bar{w}} \lambda_n$. Then $\|D\psi_{\bar{w}}^\zeta|_D\|_\infty < \gamma^n$ ($\|D\psi_{\bar{w}}^\zeta|_E\|_\infty < \gamma^n$, respectively).*

Proof. By Theorem 2.1, there exists $N > 0$ and a pair of domains $\hat{D} \ni D$ and $\hat{E} \ni E$ such that for all $n \geq N$ the maps of the pair $\mathcal{R}^n \zeta \in \mathbf{B}^{\hat{D}, \hat{E}}$. By Koebe Distortion Theorem, this implies that for all $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) the branches $\zeta^{\bar{w}}$ have bounded distortion. The claims readily follow. \square

2.1. Expansion of renormalization. In this section we will describe the expanding direction of renormalization. For the remainder of this chapter, let us fix the domains D , and E as in Theorem 1.9.

Definition of the expanding cone field. We begin by defining a subset \mathcal{C} in the tangent bundle $\mathbf{T} \equiv T\mathbf{B}^{D,E}$ as follows. Let $\bar{v}(x) \in \mathbf{T}_\zeta$ for some renormalizable pair ζ . Let ζ be a twice renormalizable pair, and recall that $p\mathcal{R}^2\zeta$ denotes the second pre-renormalization (the non-rescaled iterate) of ζ . Denote

$$\mathcal{C}_\zeta = \{\bar{v} \in \mathbf{T} \mid \inf_x \nabla_{\bar{v}} p\mathcal{R}^2\zeta > 0 \text{ for all } x \in I_2 \cup J_2\},$$

and set $\mathcal{C} = \cup \mathcal{C}_\zeta$ over all twice-renormalizable pairs $\zeta \in \mathbf{B}^{D,E}$.

Proposition 2.3. *For every twice-renormalizable ζ , the set \mathcal{C}_ζ is an open cone in \mathbf{T}_ζ .*

We next prove:

Proposition 2.4. *Let $\zeta(t) : (0, 1) \rightarrow \mathbf{B}^{D,E}$ be a smooth curve with the property*

$$\frac{d}{dt}\bar{\zeta}(t) \in \mathcal{C}_{\zeta(t)} \text{ for all } t.$$

Then the function

$$\rho(t) \equiv \rho(\zeta(t))$$

is non-decreasing. Furthermore, if $\rho(t_0) \notin \mathbb{Q}$ then $\rho(t)$ is strictly increasing at t_0 .

Proof. Fix $t_0 \in (0, 1)$ and let $\zeta(t_0)^k(0) \neq 0$ be a closest return of 0 under the dynamics of the pair $\zeta(t_0)$. An easy induction based on the Chain Rule shows that $\frac{d}{dt}\zeta(t)^k(0)|_{t=t_0}$ is positive for all k starting with the first returns corresponding to the second renormalization. Thus, the heights r_{2i} of renormalizations $\mathcal{R}^{2i}\zeta(t)$ decrease, and the heights r_{2i+1} of renormalizations $\mathcal{R}^{2i+1}\zeta(t)$ increase with t . Hence, the value of the rotation number $\rho = [r_0, r_1, \dots]$ is a non-decreasing function of t . The last assertion is similarly evident and is left to the reader. \square

The expansion properties of the cone field \mathcal{C} . We begin by recalling how the composition operator acts on vector fields. For a pair of analytic functions f and g of the real variable, denote

$$\text{Comp}(f, g) = f \circ g.$$

Consider Comp as an operator $C^\omega \times C^\omega \rightarrow C^\omega$ and let $D\text{Comp}$ denote its differential. An elementary calculation shows that

$$D\text{Comp}|_{(f,g)} : (\phi, \gamma) \rightarrow f' \circ g \cdot \gamma + \phi \circ g. \quad (2.1)$$

The significance of the formula (2.1) for us lies in the following trivial observation: if f and g are both increasing functions, and the vector fields ϕ and γ are non-negative, then

$$\inf_x D\text{Comp}|_{(f,g)}(\phi, \gamma) \geq \inf_x \phi. \quad (2.2)$$

Proposition 2.5. *Fix a twice-renormalizable pair $\zeta = (\eta, \xi) \in \mathbf{B}^{D,E}$. Then \mathcal{C}_ζ is non-empty.*

Proof. Let $\bar{v} = (\bar{\alpha}, \bar{\beta})$ have the properties:

- $\bar{\alpha}(x) > 0$, $\bar{\beta}(x) > 0$ for real x such that $x \notin \{0, 1, \eta(0)\}$;
- for each $x \in \{0, 1, \eta(0)\}$, the vector field $\bar{v}(x)$ vanishes to order 3.

It is evident that vector fields with these properties exist (they can be taken to be polynomial, for instance), and that every such $\bar{v} \in \mathbf{T}_\zeta$. Finally, $\bar{v} \in \mathcal{C}_\zeta$ by the Chain Rule (2.1). \square

For a renormalizable pair $\zeta = (\eta, \xi)$ let us set

$$\lambda_\zeta = \eta^{r_0}(1) > 0,$$

where, as before, r_i denotes the height of $\mathcal{R}^i \zeta$.

Proposition 2.6. *There exist $k \in \mathbb{N}$ and $\delta > 0$ such that the following holds. Let $\zeta \in \mathbf{B}^{D,E}$ and let $\bar{v} \in \mathcal{C}_\zeta$. Then*

$$\|D\mathcal{R}_\zeta^{2k} \bar{v}\| > C \cdot \epsilon(1 + \delta)^k,$$

where C is bounded on compact subsets of $\mathbf{B}^{D,E}$ and $\epsilon = \inf Dp\mathcal{R}^2 \bar{v}(x) > 0$.

Proof. Let $\bar{v}(x) = (\bar{\alpha}(x), \bar{\beta}(x)) \in \mathcal{C}_\zeta$. Consider a smooth deformation

$$\zeta_t^{\bar{v}} = (\eta + t\bar{\alpha} + o(t), \xi + t\bar{\beta} + o(t)) \equiv (\eta_t, \xi_t) \in \mathbf{B}^{D,E}. \quad (2.3)$$

For $m \in \mathbb{N}$ let us denote

$$\mathcal{R}^{2m} \zeta_t^{\bar{v}} \equiv (\eta_{t,m}, \xi_{t,m}), \text{ and } p\mathcal{R}^{2m} \zeta_t^{\bar{v}} \equiv (H_{t,m}, K_{t,m}).$$

Let

$$\lambda_{t,m} \equiv K_{t,m}(0) > 0.$$

An easy induction shows that

- (a) $\eta_{t,k}(x) = \frac{1}{\lambda_{t,k}} H_{t,k} \circ (\lambda_{t,k} x)$;
- (b) $H_{t,k}(0) < 0$.

A repeated application of (2.1) implies that

- (c) $\frac{\partial}{\partial t} H_{t,k}(x) > \epsilon > 0$ where $\epsilon = \inf Dp\mathcal{R}^2 \zeta^{\bar{v}}(x)$;
- (d) $\frac{d}{dt} \lambda_{t,k} > 0$.

We calculate:

$$\frac{\partial}{\partial t} \left(\frac{1}{\lambda_{t,k}} H_{t,k}(\lambda_{t,k} x) \right) = -\frac{\frac{d}{dt} \lambda_{t,k}}{(\lambda_{t,k})^2} H_{t,k}(\lambda_{t,k} x) + \frac{1}{\lambda_{t,k}} \left(\frac{\partial H_{t,k}(\lambda_{t,k} x)}{\partial t} + \frac{\partial H_{t,k}(x)}{\partial x} \frac{d\lambda_{t,k}}{dt} x \right).$$

Substituting $x = 0$ and using (a) – (d) we see that

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \left(\frac{1}{\lambda_{t,k}} H_{t,k}(\lambda_{t,k} x) \right) \Big|_{x=0} = D\mathcal{R}^{2k} \bar{v}(0) \geq \frac{1}{\lambda_{0,k}} \epsilon.$$

The standard real *a priori* bounds imply that

$$\lambda_{0,k} \leq C(1 + \delta)^{-k},$$

where $\delta > 0$ is universal, and C is bounded on compact subsets of C^3 -commuting pairs, which completes the proof. \square

2.2. Local stable manifold of a periodic point of \mathcal{R} . As before, let us work in the notation of Theorem 2.1. Set $\zeta \equiv \zeta_*$.

Set $\rho = \rho(\zeta)$, and define

$$\mathcal{D}_\rho = \{\gamma \in \mathbf{B}^{D,E}, \text{ such that } \rho(\gamma) = \rho\}.$$

The following proposition directly follows from Theorem 1.4 and compactness considerations:

Proposition 2.7. *There exists a neighborhood Y of ζ in $\mathbf{B}^{D,E}$ such that for every $\gamma \in Y \cap \mathcal{D}_\rho$*

$$\mathcal{R}^{pm}\gamma \xrightarrow{m \rightarrow \infty} \zeta$$

at a geometric rate, uniformly in Y .

Below we shall demonstrate that the local stable set of ζ is a graph over a hyperplane:

Theorem 2.8. *There is an open neighborhood $W \subset \mathbf{B}^{D,E}$ of ζ such that $\mathcal{D}_\rho \cap W$ is a C^0 -graph over a hyperplane in a local chart in $\mathbf{B}^{D,E}$.*

Denote p_k/q_k the reduced form of the k -th continued fraction convergent of ρ . Furthermore, define \mathcal{D}_k as the set of $\gamma \in \mathbf{B}^{D,E}$ for which $\rho(\gamma) = p_k/q_k$ and 0 is a periodic point with period q_k . As follows from the Implicit Function Theorem, this is a local codimension 1 submanifold. We note:

Lemma 2.9. *Let $\gamma \in \mathcal{D}_k$ for $k = 2m \geq 2$, and denote $T_\gamma \mathcal{D}_k \subset \mathbf{T}_\gamma$ the tangent space to \mathcal{D}_k at this point. Then*

$$T_\gamma \mathcal{D}_k \cap \mathcal{C}_\gamma = \emptyset.$$

Proof. Let $\bar{v} \in \mathcal{C}_\gamma$ and suppose $\{\gamma_t\}$ is a one-parameter family such that

$$\gamma_t = \gamma + t\bar{v} + o(t).$$

Then for sufficiently small values of $t > 0$, $\gamma_t^{q_k} > \gamma^{q_k}$, and hence $\gamma_t^{q_k}(0) \neq 0$. \square

Now let $\bar{v} \in \mathcal{C}_\zeta$ be as in the proof of Proposition 2.5, and let $\{\zeta_t\}$ be a one-parameter family in $\mathbf{B}^{D,E}$ such that

$$\zeta_t = \zeta + t\bar{v}.$$

Elementary considerations of the Intermediate Value Theorem imply that for every large enough k there exists a value of $t > 0$ such that the map $\zeta_t \in \mathcal{D}_k$. Moreover, if we denote t_k the smallest parameter with this property, then $t_k \rightarrow 0$. Set $\zeta_k = \zeta_{t_k}$ and let $T_k = T_{\zeta_k} \mathcal{D}_k \subset \mathbf{T}$. By Lemma 2.9 and the Hahn-Banach Theorem there exists $\epsilon > 0$ such that for every k there exists a linear functional $h_k \in (\mathbf{T}_{\zeta})^*$ with $\|h_k\| = 1$, such that $\text{Ker } h_k = T_k$ and $h_k(\bar{v}) > \epsilon$. By the Alaoglu Theorem, we may select a subsequence h_{n_k} weakly-* converging to $h \in (\mathbf{T}_\zeta)^*$. Necessarily $\bar{v} \notin \text{Ker } h$, so $h \neq 0$. Set $T = \text{Ker } h$.

Proof of Theorem 2.8. By the above, we may select a splitting $\mathbf{T}_\zeta = T \oplus \bar{v} \cdot \mathbb{R}$. Denote $p : \mathbf{T}_\zeta \rightarrow T$ the corresponding projection, and let $\psi : \mathbf{B}^{D,E} \rightarrow \mathbf{T}_\zeta$ be a local chart at ζ . Lemma 2.9 together with the Intermediate Value Theorem imply that $p \circ \psi : \mathcal{D}_k \rightarrow T$ is

an isomorphism onto the image, and there exists an open neighborhood \mathcal{U} of the origin in T , such that $p \circ \Psi(\mathcal{D}_k) \supset \mathcal{U}$. We may select a C^0 -converging subsequence \mathcal{D}_{k_j} , whose limit is a graph G over \mathcal{U} . Necessarily, for every $\gamma \in G$, $\rho(\gamma) = \rho$. As we have seen above, every point $\gamma \in \mathcal{D}_\rho$ in a sufficiently small neighborhood of ζ is in G , and thus G is an open neighborhood in \mathcal{D}_ρ . \square

2.3. Proof of Theorem 2.1. Let us work in the notation of Theorem 2.1 again. Note that by Theorem 1.9, the operator \mathcal{L} is compact, and hence, by the standard facts of the spectral theory of compact operators, we have:

- every element of the spectrum of \mathcal{L} is an eigenvalue;
- the spectrum of \mathcal{L} has no accumulation points except for 0.

Proposition 2.10. *The operator \mathcal{L} has no eigenvalues on the unit circle.*

Proof. Assume the contrary. Then, by the Central Manifold Theorem, there exists a finite-dimensional smooth central manifold at ζ which is transverse simultaneously to D_ρ at ζ and to the cone \mathcal{C}_ζ . This is clearly impossible by dimensionality considerations. \square

We now prove:

Proposition 2.11. *The operator \mathcal{L} has a single unstable eigenvalue.*

Proof. By Proposition 2.6, the operator \mathcal{L} has at least one unstable eigenvalue. On the other hand, by Theorem 2.8, $\text{codim } W^s(\zeta) < 2$. \square

3. EXTENDING RENORMALIZATION TO DISSIPATIVE TWO-DIMENSIONAL PAIRS

Fix $R > 0$ and let $\Omega = D \times \mathbb{D}_R$, $\Gamma = E \times \mathbb{D}_R$, where D and E are as in Theorem 2.1. The space of pairs maps of \mathbb{C}^2 analytic on $\Omega \times \Gamma$ will be denoted $\mathbf{U}^{\Omega, \Gamma}$. We set

$$\mathbf{O}^{\Omega, \Gamma} \equiv (\mathbf{U}^{\Omega, \Gamma})^{\mathbb{R}}.$$

Given $C > 0$, let $\mathbf{B}_C^{D, E}$ be the open subset of $\mathbf{B}^{D, E}$ such that $\eta^{-1}(0) > C$.

Let us denote

$$\mathbf{C}^{D, E} \equiv (C^\omega(D) \times C^\omega(E))^{\mathbb{R}}.$$

Given $0 < \delta < C$, consider the subset of $\mathbf{O}^{\Omega, \Gamma}$ of pairs (A, B)

$$A(x, y) = (\eta(x), h(x)), \tag{3.1}$$

$$B(x, y) = (\xi(x), g(x)), \tag{3.2}$$

where $(\eta, \xi) \in \mathbf{B}_C^{D, E}$, and

$$|\partial_x h(x)| > 0, \text{ for all } x \in D \setminus \mathbb{D}_\delta(0), \text{ and } |\partial_x g(x)| > 0 \text{ for all } x \in E \setminus \mathbb{D}_\delta(0), \tag{3.3}$$

$$|h(x)| < R/2, \text{ for all } x \in D, \text{ and } |g(x)| < R/2, \text{ for all } x \in E. \tag{3.4}$$

This subset of $\mathbf{O}^{\Omega, \Gamma}$ will be denoted $\hat{\mathbf{O}}^{\Omega, \Gamma}$. Notice that $\hat{\mathbf{O}}^{\Omega, \Gamma}$ is not relatively open in $\mathbf{O}^{\Omega, \Gamma}$. Denote $\mathbf{O}_n^{\Omega, \Gamma}$ the subset of $\mathbf{O}^{\Omega, \Gamma}$ such that the pair (η, ξ) from (3.1)-(3.2) is at least n times renormalizable, and similarly for $\hat{\mathbf{O}}_n^{\Omega, \Gamma}$.

Given a subset \mathbf{Q} of a Banach manifold \mathbf{O} , we denote \mathbf{Q}_ϵ the open subset of all elements of \mathbf{O} whose distance to \mathbf{Q} is less than ϵ .

In particular, in what follows, we will extend the definition of renormalization to the Banach manifold $\hat{\mathbf{O}}_\epsilon^{\Omega, \Gamma}$, whose elements will be denoted by

$$A(x, y) = (a(x, y), h(x, y)) = (a_y(x), h_y(x)), \quad (3.5)$$

$$B(x, y) = (b(x, y), g(x, y)) = (b_y(x), g_y(x)), \quad (3.6)$$

Let us further define a space $\mathbf{A}^{D, E}$ as the collection of pairs of $\eta \in C^\omega$, $\xi \in C^\omega$ which have bounded analytic continuations to D and E respectively, and such that

$$\eta \circ \xi(z) - \xi \circ \eta(z) = o(z^3) \text{ and } \xi(0) = 1. \quad (3.7)$$

We view $\mathbf{A}^{D, E}$ as a subset of the Banach space $\mathbf{C}^{D, E}$. Then

Proposition 3.1. *There exists ϵ_0 such that*

$$\mathbf{A}_0^{D, E} \equiv \mathbf{A}^{D, E} \cap \mathbf{B}_\epsilon^{D, E}$$

is a Banach submanifold of the real slice of $C^\omega(D) \times C^\omega(E)$. Moreover, $\mathbf{B}^{D, E}$ is a submanifold of $\mathbf{A}_0^{D, E}$ whose codimension is equal to two.

Proof. The first statement is a straightforward generalization of Proposition 1.5 and will be left to the reader. To prove the second statement, note that $\mathbf{B}^{D, E}$ is defined inside of $\mathbf{A}_0^{D, E}$ by the conditions

$$\eta'(0) = \eta''(0) = 0,$$

since the corresponding conditions for ξ follow automatically from (3.7). \square

We will define an embedding ι of the manifold $\mathbf{C}^{D, E}$ into $\hat{\mathbf{O}}^{\Omega, \Gamma}$ which sends a pair $\zeta = (\eta, \xi)$ to a pair of functions $\iota(\zeta)$ given by

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \eta(x) \\ \eta(y) \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \xi(x) \\ \xi(y) \end{pmatrix} \right).$$

Let us denote π_1 and π_2 the two coordinate projections $\mathbb{C}^2 \rightarrow \mathbb{C}$. For a pair of two-dimensional maps $(A, B)(x, y)$ let us define

$$\mathcal{L}(A, B)(x, y) \equiv (\pi_1(A(x, 0)), \pi_1(B(x, 0))) = (a(x, 0), b(x, 0)).$$

3.1. Reduction of the size of perturbation. We will now define a renormalization of operator on pairs of maps in $\hat{\mathbf{O}}_\epsilon^{\Omega, \Gamma}$ in several steps.

Similarly to [dCLM], our first step will be to define a sufficiently high prerenormalization as a pair defined in a neighborhood of $(\eta^{-1}(0), 0)$, and then pull such pair back to neighborhood of $(0, 0)$ by a coordinate change which reduces the size of the y -dependent parts of the maps.

Let $\zeta \in \mathbf{B}^{D, E}$ be n -times renormalizable, $n \geq 3$:

$$\mathcal{R}^n \zeta = \lambda_n^{-1} \circ \left(\zeta^{\bar{s}_n}, \zeta^{\bar{t}_n} \right) \circ \lambda_n,$$

and denote

$$\begin{aligned}\hat{s}_n &= \begin{cases} (a_1, b_1, a_2, b_2, a_n - 2, b_n), & a_n \geq 2 \\ (a_1, b_1, a_2, b_2, 0, b_n - 1), & a_n = 1 \end{cases}, \\ \phi_0(x) &= \begin{cases} \eta^2, & a_n \geq 2 \\ \eta \circ \xi, & a_n = 1 \end{cases}.\end{aligned}$$

Define \hat{t}_n in a similar way. Then $\mathcal{R}^n \zeta$ can be written as

$$\mathcal{R}^n \zeta = \lambda_n^{-1} \circ \phi_0 \circ (\zeta^{\hat{s}_n}, \zeta^{\hat{t}_n}) \circ \lambda_n.$$

For sufficiently large n , the function η^{-1} is a diffeomorphism of a neighborhood of $\lambda_n(D \cup E)$, while $\eta^{-1}(\lambda_n(D \cup E)) \not\subset \mathbb{D}_\delta(0)$. Define the n -th prerenormalization of ζ on $\eta^{-1}(\lambda_n(D \cup E))$ as

$$\hat{p}\mathcal{R}^n \zeta = (\eta^{-1} \circ \zeta^{\bar{s}_n} \circ \eta, \eta^{-1} \circ \zeta^{\bar{t}_n} \circ \eta) = (f \circ \zeta^{\hat{s}_n} \circ \eta, f \circ \zeta^{\hat{t}_n} \circ \eta),$$

where

$$f = \eta \text{ if } a_n \geq 2 \text{ and } f = \xi \text{ if } a_n = 1. \quad (3.8)$$

Next, suppose that ϵ is sufficiently small, so that the following prerenormalization of the pair $Z = (A, B) \in \bar{\mathbf{O}}_\epsilon^{\Omega, \Gamma}$ is defined in a neighborhood of $\eta^{-1}(\lambda_n(D \cup E))$ in \mathbb{C}^2 :

$$\hat{p}\mathcal{R}^n Z = (F \circ Z^{\hat{s}_n} \circ A, F \circ Z^{\hat{t}_n} \circ A),$$

where $F = A$ if $a_n \geq 2$ and $F = B$ if $a_n = 1$.

Set

$$\phi_y(x) = \phi(x, y) := \begin{cases} \pi_1 A^2(x, y) = a(a(x, y), h(x, y)), & a_n \geq 2 \\ \pi_1 A \circ B(x, y) = a(b(x, y), g(x, y)), & a_n = 1 \end{cases}$$

For sufficiently small ϵ , the map ϕ_z is close to ϕ_0 and is a diffeomorphism of a neighborhood of $\pi_1 Z^{\hat{s}_n}(\lambda_n(D), 0) \approx \zeta^{\hat{s}_n}(\lambda_n(D))$ for all $z \in \mathbb{D}_R$ for some $R = R(\epsilon) > 0$. Similarly, g_z is a diffeomorphism of a neighborhood of $\pi_1 Z^{\hat{s}_n}(\lambda_n(Z), 0)$ for all $z \in \mathbb{D}_R$ for some $R = R(\epsilon) > 0$.

Furthermore, set

$$q_z(x) \equiv q(x, z) = \pi_2 F(x, z) = \begin{cases} h_z(x), & a_n \geq 2 \\ g_z(x), & a_n = 1 \end{cases} \quad (3.9)$$

According to (3.3), this is a diffeomorphism outside a neighborhood of zero. Also, set

$$w_z(x) \equiv w(x, z) := q_z(\phi_z^{-1}(x)),$$

a diffeomorphism of a neighborhood of $\pi_1 \phi_z \circ Z^{\hat{s}_n}(\lambda_n(D), 0)$ in \mathbb{C}^2 onto its image for all $z \in \mathbb{D}_R$ for some $R = R(\epsilon) > 0$. Notice, that $\partial_z w_z(x)$ and $\partial_z w_z^{-1}(x)$ are functions whose uniform norms are $O(\epsilon)$ on compact subsets of their domains of definition. Notice, that

$$w_z^{-1}(x) := \phi_z(q_z^{-1}(x)) = \begin{cases} a(a_z(h_z^{-1}(x)), x), & a_n \geq 2 \\ a(b_z(g_z^{-1}(x)), x), & a_n = 1 \end{cases}$$

is close to $\eta(\eta(h^{-1}(x)))$, $a_n \geq 2$, or $\eta(\xi(g^{-1}(x)))$, $a_n = 1$, and is a diffeomorphism outside a neighborhood of $h(0)$ or $g(0)$, respectively.

We are now ready to define the first change of coordinates:

$$H(x, y) = (a_y(x), w_{q_0^{-1}(y)}^{-1}(y)), \quad (3.10)$$

where the function g_z^{-1} is defined by

$$q_z^{-1}(q(x, z)) = x \quad (3.11)$$

This transformation is ϵ -close to $(\eta(x), \phi_0(q_0^{-1}(y)))$ in $\hat{\mathbf{O}}_\epsilon^{\Omega, \Gamma}$, and therefore, for small ϵ , is a diffeomorphism of a neighborhood of $\pi_1 F \circ Z^{\hat{s}_n}(\lambda_n(D), 0) \approx f(\zeta^{\hat{s}_n}(\lambda_n(D)))$ onto its image.

Proposition 3.2. *If ϵ is sufficiently small and n is sufficiently large, then the following holds:*

- a) *the inverse H^{-1} is a well defined analytic diffeomorphism on $\Lambda_n(\Omega \cup \Gamma)$, where $\Omega = D \times \mathbb{D}_R$ and $\Gamma = E \times \mathbb{D}_R$,*
- b) *$p\mathcal{R}^n Z$ is a pair of well-defined analytic transformations on $\Lambda_n(\Omega \cup \Gamma)$.*

Proof. We have

$$\begin{aligned} H^{-1}(x, y) &= \left(a_{w_{\phi_0^{-1}(y)}^{-1}(y)}^{-1}(x), w_{\phi_0^{-1}(y)}^{-1}(y) \right) + O(\epsilon^2) \\ &= (\eta^{-1}(x), q_0(\phi_0^{-1}(y))) + O(\epsilon). \end{aligned}$$

Thus, if n is large enough, then, according to (3.3), H^{-1} is a well-defined analytic diffeomorphism of $\Lambda_n(\Omega \cup \Gamma)$.

Furthermore,

$$\begin{aligned} A \circ H^{-1}(x, y) &= \left(x, h(a_{\beta(x, y)}^{-1}(x), \beta(y)) \right) + O(\epsilon^2) \\ &= (x, h(\eta^{-1}(x), 0)) + O(\epsilon), \quad \beta(y) := w_{\phi_0^{-1}(y)}^{-1}(y). \end{aligned}$$

By (3.4), if ϵ is sufficiently small, then $|h(\eta^{-1}(x), 0)| < 2R/3$, and $A \circ H^{-1}$ maps $\Lambda_n(\Omega \cup \Gamma)$ into an $O(\epsilon)$ -neighborhood of the set $\Lambda_n((D \times \mathbb{D}_{\frac{2}{3}R}) \cup (E \times \mathbb{D}_{\frac{2}{3}R}))$.

If ϵ is small enough, the branches $F \circ Z^{\hat{s}_n}$ and $F \circ Z^{\hat{t}_n}$ are well-defined on $O(\epsilon)$ -neighborhoods of $\Lambda_n(D \times \mathbb{D}_{\frac{2}{3}R})$ and $\Lambda_n(E \times \mathbb{D}_{\frac{2}{3}R})$, respectively, and map these neighborhoods to an $O(\epsilon)$ -neighborhood of $f \circ \zeta^{\hat{s}_n}(\lambda_n(D)) \times q_0 \circ \zeta^{\hat{s}_n}(\lambda_n(D))$ and $f \circ \zeta^{\hat{t}_n}(\lambda_n(E)) \times q_0 \circ \zeta^{\hat{t}_n}(\lambda_n(E))$, where f is defined in (3.8) and q in (3.9). Again, by (3.4),

$$\left| q \circ Z^{\hat{s}_n}(\Lambda_n(D \times \mathbb{D}_{\frac{2}{3}R})) \right| < \frac{2}{3}R, \text{ and } \left| q \circ Z^{\hat{t}_n}(\Lambda_n(E \times \mathbb{D}_{\frac{2}{3}R})) \right| < \frac{2}{3}R,$$

if n is large enough and ϵ is small. The second claim readily follows. \square

Notice that

$$w_{g_0^{-1}(y)}^{-1}(y) - w_{g_z^{-1}(y)}^{-1}(y) \sim \left(\partial_z w_{g_z^{-1}(y)}^{-1}(y) \right) (\partial_z g_z^{-1}(y)) z = O(\delta^2)O(z), \quad (3.12)$$

$$w_{z_1}^{-1} \circ w_{z_2} - id \sim \partial_z w_z(z_2 - z_1) = O(\delta)O(z_2 - z_1). \quad (3.13)$$

We use $H(x, y)$ to pull back $\hat{p}\mathcal{R}^n Z$ to a neighborhood of definition of the n -th prerenormalization of an a.c. pair (η, ξ) - that is, a neighborhood of $\lambda_n(D \cup E)$ in \mathbb{C}^2 :

$$p\mathcal{R}^n Z = (\bar{A}, \bar{B}) = H \circ F \circ \left(Z^{\hat{s}_n}, Z^{\hat{t}_n} \right) \circ A \circ H^{-1}(x, y).$$

The first map in this pair

$$\begin{aligned} \bar{A}(x, y) &= H \circ F \circ Z^{\hat{s}_n} \circ A \circ H^{-1}(x, y) \\ &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ w_{q_0^{-1}(\pi_2 F \circ Z^{\hat{s}_n})}^{-1} \circ \pi_2 F \circ Z^{\hat{s}_n} \end{array} \right) \circ A \circ H^{-1}(x, y) \\ &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ w_{(q_0^{-1} \circ q) \circ Z^{\hat{s}_n}}^{-1} \circ \pi_2 F \circ Z^{\hat{s}_n} \end{array} \right) \circ A \circ H^{-1}(x, y) \\ (3.12) \quad &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ w_{(q_{\pi_2 Z^{\hat{s}_n}}^{-1} \circ q) \circ Z^{\hat{s}_n}}^{-1} \circ \pi_2 F \circ Z^{\hat{s}_n} + O(\epsilon^2) \end{array} \right) \circ A \circ H^{-1}(x, y) \\ (3.11) \quad &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ w_{\pi_1 Z^{\hat{s}_n}}^{-1} \circ q_{\pi_2 Z^{\hat{s}_n}} \circ \pi_1 Z^{\hat{s}_n} + O(\epsilon^2) \end{array} \right) \circ A \circ H^{-1}(x, y) \\ &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ w_{\pi_1 Z^{\hat{s}_n}}^{-1} \circ w_{\pi_2 Z^{\hat{s}_n}} \circ \phi_{\pi_2 Z^{\hat{s}_n}} \circ \pi_1 Z^{\hat{s}_n} + O(\epsilon^2) \end{array} \right) \circ A \circ H^{-1}(x, y) \\ (3.13) \quad &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ \pi_1 A \circ F \circ Z^{\hat{s}_n} + O(\epsilon)O(\pi_1 Z^{\hat{s}_n} - \pi_2 Z^{\hat{s}_n}) + O(\epsilon^2) \end{array} \right) \circ A \circ H^{-1}(x, y). \end{aligned}$$

We have

$$\begin{aligned} \bar{A}(x, y) &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ \pi_1 A \circ F \circ Z^{\hat{s}_n} + O(\epsilon)O(\pi_1 Z^{\hat{s}_n} - \pi_2 Z^{\hat{s}_n}) \end{array} \right) (x, h(a_{\beta(y)}^{-1}(x), \beta(y)) + O(\epsilon^2)) \\ &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{s}_n} \\ \pi_1 A \circ F \circ Z^{\hat{s}_n} + O(\epsilon)O(\pi_1 Z^{\hat{s}_n} - \pi_2 Z^{\hat{s}_n}) \end{array} \right) (x, h(a_{\beta(0)}^{-1}(x), \beta(0)) + O(\epsilon^2)). \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{B}(x, y) &= H \circ F \circ Z^{\hat{t}_n} \circ A \circ H^{-1}(x, y) \\ &= \left(\begin{array}{c} \pi_1 A \circ F \circ Z^{\hat{t}_n} \\ \pi_1 A \circ F \circ Z^{\hat{t}_n} + O(\epsilon)O(\pi_1 Z^{\hat{t}_n} - \pi_2 Z^{\hat{t}_n}) \end{array} \right) (x, h(a_{\beta(0)}^{-1}(x), \beta(0)) + O(\epsilon^2)). \end{aligned}$$

In particular the pair (\bar{A}, \bar{B}) is $O(\epsilon(\|\pi_1 Z^{\hat{t}_n} - \pi_2 Z^{\hat{t}_n}\| + \epsilon))$ -close to $\iota(\mathbf{C}^{D,E})$.

We will also write

$$\bar{A}(x, y) = \begin{pmatrix} \bar{\eta}_1(x) + \bar{\tau}(x, y) \\ \bar{\eta}_2(x) + \bar{\tau}_2(x, y) \end{pmatrix}, \quad (3.14)$$

where

$$\bar{\eta}_1(x) \equiv \pi_1 \bar{A}(x, 0), \quad \bar{\eta}_2(x) \equiv \pi_2 \bar{A}(x, 0)$$

are $O(\epsilon(\|\pi_1 Z^{\hat{s}_n} - \pi_2 Z^{\hat{s}_n}\| + \epsilon))$ -close to each other, and both are ϵ -close to $\pi_{\eta} p \mathcal{R}^n \zeta = \phi_0 \circ \zeta^{\hat{s}_n} = \zeta^{\bar{s}_n}$, and

$$\bar{\tau}_1(x, y) \equiv \pi_1 \bar{A}(x, y) - \pi_1 \bar{A}(x, 0), \quad \bar{\tau}_2(x, y) \equiv \pi_2 \bar{A}(x, y) - \pi_2 \bar{A}(x, 0),$$

are functions whose norms are $O(\epsilon^2)$. Similarly,

$$\bar{B}(x, y) = \begin{pmatrix} \bar{\xi}_1(x) + \bar{\pi}_1(x, y) \\ \bar{\xi}_2(x) + \bar{\pi}_2(x, y) \end{pmatrix},$$

where

$$\bar{\xi}_1(x) \equiv \pi_1 \bar{B}(x, 0), \quad \bar{\xi}_2(x) \equiv \pi_2 \bar{B}(x, 0)$$

are $O(\epsilon(\|\pi_1 Z^{\hat{t}_n} - \pi_2 Z^{\hat{t}_n}\| + \epsilon))$ -close to each other, and both are ϵ -close to $\pi_{\xi} p \mathcal{R}^n \zeta = \phi_0 \circ \zeta^{\hat{t}_n} = \zeta^{\bar{t}_n}$, and

$$\bar{\pi}_1(x, y) \equiv \pi_1 \bar{B}(x, y) - \pi_1 \bar{B}(x, 0), \quad \bar{\pi}_2(x, y) \equiv \pi_2 \bar{B}(x, y) - \pi_2 \bar{B}(x, 0),$$

are functions whose norms are $O(\epsilon^2)$.

We have the following

Lemma 3.3. *There exists an $n \in \mathbb{N}$, $\epsilon > 0$, and $K > 0$, such that every $Z \in (\hat{\mathbf{O}}_{2n}^{\Omega, \Gamma})_{\epsilon}$ is n -times prerenormalizable, and*

$$p \mathcal{R}^n Z \in \iota(\mathbf{C}^{\lambda_n(D), \lambda_n(E)})_{K\epsilon(\|\pi_1 Z - \pi_2 Z\| + \epsilon)}.$$

3.2. Projection on the almost commuting pairs. At the next step we will project the pair (\tilde{A}, \tilde{B}) onto the subset of $\mathbf{O}^{\Omega, \Gamma}$ given by the conditions

$$\pi_1(\tilde{A} \circ \tilde{B}(x, 0) - \tilde{B} \circ \tilde{A}(x, 0)) = o(|x|^3), \quad (3.15)$$

$$\pi_1 \tilde{B}(0, 0) = 1. \quad (3.16)$$

To that end we set

$$\Pi(\tilde{A}, \tilde{B})(x, y) = \left(\begin{pmatrix} \tilde{\eta}_1(x) + ax^4 + bx^6 + \tilde{\tau}_1(x, y) \\ \tilde{\eta}_2(x) + ax^4 + bx^6 + \tilde{\tau}_2(x, y) \end{pmatrix}, \begin{pmatrix} \tilde{\xi}_1(x) + c + dx + ex^2 + \tilde{\pi}_1(x, y) \\ \tilde{\xi}_2(x) + c + dx + ex^2 + \tilde{\pi}_2(x, y) \end{pmatrix} \right),$$

and require that (3.15) and (3.16) are satisfied for maps in the pair $\Pi(\tilde{A}, \tilde{B})(x, y)$.

Proposition 3.4. *There exist $\epsilon > 0$, $L > 0$, such that the following holds. For every pair $(A, B) \in \iota(\mathbf{B}^{D, E})_{\epsilon}$ there exists a unique tuple $(a, b, c, d, e) \in \mathbb{D}_{L\epsilon^2}(0)^{\otimes 5}$ so that the conditions (3.15)-(3.16) hold. Furthermore, the map*

$$(A, B) \mapsto (a, b, d, e, c)$$

is analytic.

Proof. Consider the following system of 5 equations $\mathbf{F}(a, b, d, e, c) = 0$:

$$\begin{aligned} a &+ b - d\tilde{\eta}_1(0) - e\tilde{\eta}_1(0)^2 - c - \left(\tilde{\eta}_1(\tilde{\xi}_1(0)) - \tilde{\eta}_1(\tilde{\xi}_1(0) + c) \right) - \\ &- \left\{ \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0)) - \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c) \right\} \\ &= \pi_1(\tilde{B} \circ \tilde{A}(0, 0) - \tilde{A} \circ \tilde{B}(0, 0)) \end{aligned}$$

$$\begin{aligned} (\tilde{\xi}'_1(0) &+ d)(4a + 6b) + \tilde{\eta}'_1(\tilde{\xi}_1(0) + c)(\tilde{\xi}'_1(0) + d) - \tilde{\eta}'_1(\tilde{\xi}_1(0))\tilde{\xi}'_1(0) + \\ &+ \tilde{\xi}'_1(\tilde{\eta}_1(0))\tilde{\eta}'_1(0) - (\tilde{\xi}'_1(\tilde{\eta}_1(0)) + d + 2e\tilde{\eta}_1(0))\tilde{\eta}'_1(0) - \\ &+ \left\{ \nabla \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c) \cdot (\tilde{\xi}'_1(0) + d, \tilde{\xi}'_2(0) + d) - \nabla \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0)) \cdot (\tilde{\xi}'_1(0), \tilde{\xi}'_2(0)) \right\} \\ &= \pi_1(\tilde{B} \circ \tilde{A}(x, 0) - \tilde{A} \circ \tilde{B}(x, 0))'|_{x=0} \end{aligned}$$

$$\begin{aligned} (\tilde{\xi}'_1(0) &+ d)^2(12a + 30b) + (\tilde{\xi}''_1(0) + 2e)(4a + 6b) + \\ &+ \tilde{\eta}''_1(\tilde{\xi}_1(0) + c)(\tilde{\xi}'_1(0) + d)^2 + \tilde{\eta}''_1(\tilde{\xi}_1(0) + c)(\tilde{\xi}''_1(0) + 2e) - \\ &- (\tilde{\xi}''_1(\tilde{\eta}_1(0)) + 2e)\tilde{\eta}'_1(0)^2 - (\tilde{\xi}'_1(\tilde{\eta}_1(0)) + d + 2e\tilde{\eta}_1(0))\tilde{\eta}''_1(0) - \\ &- \tilde{\eta}''_1(\tilde{\xi}_1(0))\tilde{\xi}'_1(0)^2 - \tilde{\eta}'_1(\tilde{\xi}_1(0))\tilde{\xi}''_1(0) + \tilde{\xi}''_1(\tilde{\eta}_1(0))\tilde{\eta}'_1(0)^2 + \tilde{\xi}'_1(\tilde{\eta}_1(0))\tilde{\eta}''_1(0) \\ &+ \left\{ \sum_{i,j=1,2} \partial_{i,j} \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c)(\tilde{\xi}'_i(0) + d)(\tilde{\xi}'_j(0) + d) + \right. \\ &+ \nabla \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c) \cdot (\tilde{\xi}''_1(0) + 2e, \tilde{\xi}''_2(0) + 2e) - \\ &- \left. \sum_{i,j=1,2} \partial_{i,j} \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0))(\tilde{\xi}'_i(0))(\tilde{\xi}'_j(0)) - \nabla \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0)) \cdot (\tilde{\xi}''_1(0), \tilde{\xi}''_2(0)) \right\} \\ &= \pi_1(\tilde{B} \circ \tilde{A}(x, 0) - \tilde{A} \circ \tilde{B}(x, 0))''|_{x=0} \end{aligned}$$

$$\begin{aligned}
& (\tilde{\xi}'_1(0) + d)(24a + 120b) + 3(\tilde{\xi}'_1(0) + d)(\tilde{\xi}''_1(0) + 2e)(12a + 30b) + \tilde{\xi}'''_1(0)(4a + 6b) + \\
& + \tilde{\eta}'''_1(\tilde{\xi}_1(0) + c)(\tilde{\xi}'_1(0) + d) - \tilde{\eta}'''_1(\tilde{\xi}_1(0))\tilde{\xi}'_1(0) + \\
& + 3\tilde{\eta}''_1(\tilde{\xi}_1(0) + c)(\tilde{\xi}'_1(0) + d)(\tilde{\xi}''_1(0) + 2e) - 3\tilde{\eta}''_1(\tilde{\xi}_1(0))\tilde{\xi}'_1(0)\tilde{\xi}''_1(0) + \\
& + \tilde{\eta}'_1(\tilde{\xi}_1(0) + c)\tilde{\xi}'''_1(0) - \tilde{\eta}'_1(\tilde{\xi}_1(0))\tilde{\xi}'''_1(0) - \\
& - 3(\tilde{\xi}''_1(\tilde{\eta}_1(0)) + 2e)\tilde{\eta}'_1(0)\tilde{\eta}''_1(0) + 3\tilde{\xi}''_1(\tilde{\eta}_1(0))\tilde{\eta}'_1(0)\tilde{\eta}''_1(0) - \\
& - (\tilde{\xi}'_1(\tilde{\eta}_1(0)) + d + 2e\tilde{\eta}_1(0))\tilde{\eta}'''_1(0) + \tilde{\xi}'_1(\tilde{\eta}_1(0))\tilde{\eta}'''_1(0) - \\
& + \left\{ \sum_{i,j,k=1,2} \partial_{i,j,k} \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c)(\tilde{\xi}'_i(0) + d)(\tilde{\xi}'_j(0) + d)(\tilde{\xi}'_k(0) + d) + \right. \\
& + 3 \sum_{i,j=1,2} \partial_{i,j} \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c)(\tilde{\xi}''_i(0) + 2e)(\tilde{\xi}'_j(0) + d) \\
& + \nabla \tilde{\tau}_1(\tilde{\xi}_1(0) + c, \tilde{\xi}_2(0) + c) \cdot (\tilde{\xi}'''_1(0), \tilde{\xi}'''_2(0)) - \sum_{i,j,k=1,2} \partial_{i,j,k} \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0))\tilde{\xi}''_i(0)\tilde{\xi}'_j(0)\tilde{\xi}''_k(0) - \\
& \left. - 3 \sum_{i,j=1,2} \partial_{i,j} \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0))\tilde{\xi}''_i(0)\tilde{\xi}'_j(0) - \nabla \tilde{\tau}_1(\tilde{\xi}_1(0), \tilde{\xi}_2(0)) \cdot (\tilde{\xi}'''_1(0), \tilde{\xi}'''_2(0)) \right\} \\
& = \pi_1(\tilde{B} \circ \tilde{A}(x, 0) - \tilde{A} \circ \tilde{B}(x, 0))''|_{x=0}
\end{aligned}$$

$$c = 1 - \tilde{\xi}_1(0).$$

The functions in the parenthesis above have the uniform norm $O(\epsilon^2) \cdot \max\{c, d, e\}$.

Notice, that when the commutator $\pi_1(A \circ B - B \circ A)(x, 0) = o(|x|^3)$ and B is normalized appropriately, $B(0, 0) = (1, 1)$, this system of equations is solved by $a = b = d = e = c = 0$. Furthermore, denote $\mathbf{p} = (a, b, d, e, c)$, then the derivative $D_{\mathbf{p}}\mathbf{F}(\mathbf{0})$ is given by

$$\begin{bmatrix}
1 & 1 & -\eta_1(0) & -\eta_1(0)^2 & a_{1,5} \\
4\varepsilon_1 & 6\varepsilon_1 & \tilde{\eta}'_1(\tilde{\xi}_1(0)) - \nu_1 + \delta_1 & -2\tilde{\eta}_1(0)\nu_1 & a_{2,5} \\
12\varepsilon_1^2 + 4\alpha_1 & 30\varepsilon_1^2 + 6\alpha_1 & 2\varepsilon_1\tilde{\eta}''_1(\tilde{\xi}_1(0)) - \beta_1 + \delta_2 & 2\tilde{\eta}'_1(\tilde{\xi}_1(0)) - 2\nu_1^2 - 2\tilde{\eta}_1(0)\beta_1 + \delta_3 & a_{3,5} \\
\frac{4\tilde{\xi}'''_1(0) + 24\varepsilon_1 + 36\varepsilon_1\alpha_1}{120\varepsilon_1 + 90\varepsilon_1\alpha_1} & \frac{6\tilde{\xi}'''_1(0) + 120\varepsilon_1 + 90\varepsilon_1\alpha_1}{120\varepsilon_1 + 90\varepsilon_1\alpha_1} & \frac{\tilde{\eta}'''_1(\tilde{\xi}_1(0)) - \tilde{\eta}'''_1(0) + 3\tilde{\eta}''_1(\tilde{\xi}_1(0))\alpha_1 + \delta_4}{+3\tilde{\eta}''_1(\tilde{\xi}_1(0))\alpha_1 + \delta_4} & \frac{-2\tilde{\eta}_1(0)\tilde{\eta}'''_1(0) - 6\tilde{\eta}''_1(\tilde{\xi}_1(0))\varepsilon_1 - 6\beta_1\nu_1 + \delta_5}{-6\tilde{\eta}''_1(\tilde{\xi}_1(0))\varepsilon_1 - 6\beta_1\nu_1 + \delta_5} & a_{4,5} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

where $a_{i,5}$ denote certain bounded numbers whose values are irrelevant for the computation of the determinant, $\varepsilon_i = \tilde{\xi}'_i(0)$, $\nu_i = \eta'_i(0)$, $\alpha_i = \tilde{\xi}''_i(0)$, $\beta_i = \tilde{\eta}''_i(0)$, $i = 1, 2$, while δ_i are some number whose size is $O(\epsilon^2)$.

The determinant of the above matrix is $\max\{\varepsilon_1, \nu_1, \alpha_1, \beta_1, \epsilon\}$ -close to $4(\tilde{\eta}'_1(\tilde{\xi}_1(0)))^2 \tilde{\xi}_1'''(0)$ and is nonzero for $(A, B) \in \iota(\mathbf{B}^{D,E})_\epsilon$ if ϵ is sufficiently small. By the Regular Value Theorem, there exists $\epsilon > 0$ such that for each $(A, B) \in \iota(\mathbf{B}^{D,E})_\epsilon$, there is a unique 5-tuple (a, b, d, e, c) which depends on (A, B) analytically, so that $\pi_1(\tilde{A} \circ \tilde{B} - \tilde{B} \circ \tilde{A})(x, 0) = o(|x|^3)$ and $\pi_1 B(0, 0) = 1$. \square

We define the *order n renormalization* of a pair $(A, B) \in \hat{\mathbf{O}}_\epsilon^{\Omega, \Gamma}$ as

$$\mathcal{R}^n(A, B) = \Pi \Lambda_n^{-1} \circ p \mathcal{R}^n(A, B) \circ \Lambda_n. \quad (3.17)$$

where $\Lambda_n(x, y) = (\ell_n x, \ell_n y)$, and $\ell_n = \pi_1 \bar{B}(0, 0)$.

According to Lemma (3.3) and Prop. (3.4), there exists $K' > 0$ such that $\mathcal{R}^n(A, B)$ is well-defined and is in $\iota(\mathbf{C}^{D,E})_{K'\epsilon(\|\pi_1 Z - \pi_2 Z\| + \epsilon)}$ if ϵ is sufficiently small and n is sufficiently large.

Let ζ_* be a periodic point of one-dimensional renormalization with period k . As a consequence of the above discussion, we have the following:

Theorem 3.5. *There exists an $n = mk \in \mathbb{N}$, $\epsilon_1 > 0$ and $K' > 0$ such that for every $\epsilon < \epsilon_1$ the following holds. The operator \mathcal{R}^n is defined and analytic on the ϵ -neighborhood $U_\epsilon(\iota(\zeta_*))$. Further, for every $Z \in U_\epsilon(\iota(\zeta_*))$, we have*

$$\mathcal{R}^n Z \in \iota(\mathbf{A}^{D,E})_{K'\epsilon(\|\pi_1 Z - \pi_2 Z\| + \epsilon)}.$$

Additionally, if Z does not depend on y then

$$\mathcal{R}^n Z = \begin{pmatrix} \mathcal{R}^n \zeta \\ \mathcal{R}^n \zeta \end{pmatrix} \in \iota(\mathbf{B}^{D,E}).$$

Hence, we have:

Theorem 3.6. *The point $Z_* = \iota(\zeta_*)$ is a fixed point of \mathcal{R}^n . The differential $\mathcal{D} = D|_{Z_*} \mathcal{R}^n$ is a compact operator. All but at most three of the eigenvalues of \mathcal{D} lie inside the open unit disk.*

Proof. According to Lemma (3.3) and Prop. (3.4),

$$\text{dist}((\mathcal{R}^n)^2 Z, \iota(\mathbf{A}^{D,E})) < C\epsilon^2.$$

The claim now follows. \square

4. ATTRACTORS OF DISSIPATIVE MAPS

As before, let $\mathcal{R}^p(\zeta_*) = \zeta_*$. Fix $\rho_* \equiv \rho(\zeta_*) \in (0, 1) \setminus \mathbb{Q}$. Set $T_a(x) \equiv x + a$, and

$$T_* \equiv (T_{\rho_*}|_{[-1,0]}, T_{-1}|_{[0,\rho_*]}).$$

The main result of this section is the following theorem:

Theorem 4.1. *Let $\zeta_* = \mathcal{R}^p(\zeta_*)$ be as above and let*

$$Z_* = (A_*, B_*) = \iota(\zeta_*) \in \mathbf{A}^{D,E,\epsilon}.$$

Suppose $Z = (A, B) \in W_{loc}^s(Z_) \subset \mathbf{A}^{D,E}$, and suppose that maps A and B commute, that is $A \circ B = B \circ A$, where defined. Then ζ has a minimal attractor Σ in $((D \cup E) \cap \mathbb{R}) \times \mathbb{R}$. Σ is a Jordan arc, and the restriction $\zeta|_\Sigma$ is topologically but not smoothly conjugate to T_* .*

Proof. Let $Z = (A, B)$.

Below, for brevity, we will denote $\Upsilon^1 = \Omega$, $\Upsilon^2 = \Gamma$.

We set $n = pm$ for some $m \geq 1$ to be fixed later. For notational simplicity, we set

$$\mathcal{R} = \mathcal{R}^n.$$

To differentiate between transformations for different pairs we will use the following notation. Given a pair Z , denote Λ_Z the rescaling that corresponds to the first renormalization \mathcal{R} , and H_Z - the transformation constructed for Z , that is

$$\mathcal{R}Z = \Lambda_Z^{-1} \circ H_Z \circ (Z^{\tilde{s}_n}, Z^{\tilde{t}_n}) \circ H_Z^{-1} \circ \Lambda_Z = \Lambda_Z^{-1} \circ H_Z \circ \hat{p}\mathcal{R}^n Z \circ H_Z^{-1} \circ \Lambda_Z$$

(since the elements of Z commute, the projections $\Pi_i = \text{Id}$).

It is instructive to note that $\mathcal{R}^l Z \neq \mathcal{R}^{ln} Z$:

$$\mathcal{R}^l Z = L_{\mathcal{R}^{l-1}Z}^{-1} \circ \dots \circ L_Z^{-1} \circ \hat{p}\mathcal{R}^{ln} Z \circ L_Z \circ \dots \circ L_{\mathcal{R}^{l-1}Z} \neq \Lambda_{ln}^{-1} \circ H_Z \circ \hat{p}\mathcal{R}^{ln} Z \circ H_Z^{-1} \circ \Lambda_{ln} = \mathcal{R}^{ln} Z.$$

For each multi-index $\bar{w} = (a_0, b_0, a_1, b_1, \dots, a_k, b_k) \prec \tilde{s}_{ln}$ or $\bar{w} = (a_1, b_1, \dots, a_k, b_k) \prec \tilde{t}_{ln}$ we define a domain

$$Q_{\bar{w}}^i = Z^{\bar{w}} \circ L_Z \circ L_{\mathcal{R}Z} \circ \dots \circ L_{\mathcal{R}^{l-1}Z}(\Upsilon^i), \quad i = 1 \text{ or } 2, \quad (4.1)$$

where

$$L_Z = H_Z^{-1} \circ \Lambda_Z.$$

By analogy with a dynamical partition of a commuting pair, the collection

$$\mathcal{Q}_{ln} \equiv \{Q_{\bar{w}}^i\}$$

will be referred to as the ln -th partition for the two-dimensional pair Z .

Given $Z \in W_{loc}^s(Z_*)$, consider the following collection of functions defined on $\Omega \cup \Gamma$:

$$\Psi_{\bar{w}}^Z = Z^{\bar{w}} \circ L_Z.$$

Given a collection of index sets $\{\bar{w}^i\}$, $\bar{w}^i \prec \bar{s}_n$ or $\bar{w}^i \prec \bar{t}_n$, consider the following *renormalization microscope*

$$\Phi_{\bar{w}^0, \bar{w}^1, \bar{w}^2, \dots, \bar{w}^{k-1}, Z}^k = \Psi_{\bar{w}^0}^Z \circ \Psi_{\bar{w}^1}^{\mathcal{R}Z} \circ \dots \circ \Psi_{\bar{w}^{k-1}}^{\mathcal{R}^{(k-1)}Z},$$

which we will also denote $\Phi_{\bar{w}_0^{k-1}, Z}^k$, $\hat{w}_0^{k-1} = \{\bar{w}^0, \bar{w}^1, \bar{w}^2, \dots, \bar{w}^{k-1}\}$, for brevity.

Lemma 4.2. *The renormalization microscope maps a set Υ^i onto an element of partition \mathcal{Q}_{kn} for Z .*

Proof. The claim holds for $k = 1$ by the definition (4.1) of the elements of the partition.

Assume that it $\Phi_{\hat{w}_0^k, Z}^k(\Upsilon^i)$ is an element of partition \mathcal{Q}_{kn} for Z .

Consider $\Phi_{\hat{w}_0^k, Z}^{k+1}(\Upsilon^i)$:

$$\Phi_{\hat{w}_0^k, Z}^{k+1}(\Upsilon^i) = \Psi_{\bar{w}^0}^Z \circ \Psi_{\bar{w}^1}^{\mathcal{R}Z} \circ \dots \circ \Psi_{\bar{w}^k}^{\mathcal{R}^k Z}(\Upsilon^i).$$

By assumption,

$$\Phi_{\hat{w}_1^k, \mathcal{R}Z}^k(\Upsilon^i) \equiv \Psi_{\bar{w}^1}^{\mathcal{R}Z} \circ \dots \circ \Psi_{\bar{w}^k}^{\mathcal{R}^k Z}(\Upsilon^i)$$

is an element of the partition of level kn for the pair $\mathcal{R}Z$, that is, by (4.1)

$$\Phi_{\hat{w}_1^k, \mathcal{R}Z}^k(\Upsilon^i) = (\mathcal{R}Z)^{\bar{v}} \circ L_{\mathcal{R}Z} \circ L_{\mathcal{R}^2 Z} \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i),$$

for some admissible $\bar{v} = (\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m)$. Therefore, using the shorthand

$$\mathcal{R}Z = (A_1, B_1),$$

we have:

$$\begin{aligned} \Phi_{\hat{w}_0^k, Z}^{k+1}(\Upsilon^i) &= \Psi_{\bar{w}^0}^Z \circ \Phi_{\hat{w}_1^k, \mathcal{R}Z}^k(\Upsilon^i), \\ &= Z^{\bar{w}^0} \circ L_Z \circ (\mathcal{R}Z)^{\bar{v}} \circ L_{\mathcal{R}Z} \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i) \\ &= Z^{\bar{w}^0} \circ L_Z \circ (B_1^{\beta_m} \circ A_1^{\alpha_m} \circ \dots \circ B_1^{\beta_0} \circ A_1^{\alpha_0}) \circ L_{\mathcal{R}Z} \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i) \\ &= Z^{\bar{w}^0} \circ L_Z \circ \Lambda_Z^{-1} \circ H_Z \circ \left((Z^{\bar{t}_n})^{\beta_m} \circ (Z^{\bar{s}_n})^{\alpha_m} \circ \dots \circ (Z^{\bar{t}_n})^{\beta_0} \circ (Z^{\bar{s}_n})^{\alpha_0} \right) \circ \\ &\quad \circ H_z^{-1} \circ \Lambda_Z \circ L_{\mathcal{R}Z} \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i) \\ &= Z^{\bar{w}^0} \circ (Z^{\bar{t}_n})^{\beta_m} \circ (Z^{\bar{s}_n})^{\alpha_m} \circ \dots \circ (Z^{\bar{t}_n})^{\beta_0} \circ (Z^{\bar{s}_n})^{\alpha_0} \circ L_Z \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i) \\ &= Z^{\bar{u}} \circ L_Z \circ \dots \circ L_{\mathcal{R}^k Z}(\Upsilon^i), \end{aligned}$$

for some index \bar{u} . By (4.1), the latter is an element of the partition $\mathcal{Q}_{(k+1)n}$. \square

Since $\mathcal{R}^l Z$ converges to Z_* at a geometric rate, the function $\Psi_{\bar{w}}^{\mathcal{R}^l Z}$ converges to the function $(\psi_{\bar{w}}^*, \psi_{\bar{w}}^{\zeta_*})$, defined in Corollary 2.2, at a geometric rate in C^1 -metric. Therefore, by Corollary 2.2, there exists a neighborhood \mathcal{S} in $W_{\text{loc}}^s(Z_*)$ of Z_* , and a sufficiently large $n = pm$, such that

$$\|D\Psi_{\bar{w}}^{\mathcal{R}^l Z}|_{\Upsilon^i}\|_{\infty} < \frac{1}{2},$$

whenever $\mathcal{R}^l Z \in \mathcal{S}$.

For every $Z \in W_{\text{loc}}^s(Z_*)$, there exists $i_0 \in \mathbb{N}$ such that $\mathcal{R}^i Z \in \mathcal{S}$ for $i \geq i_0$. Hence, there exists $C = C(Z)$, such that

$$\|D\Phi_Z^k|_{\Upsilon^i}\|_{\infty} < \frac{C}{2^k}, \quad (4.2)$$

and thus the renormalization microscope is a uniform metric contraction.

We are now ready to finish the proof.

Select a distinct point $(x_{\bar{w}}, y_{\bar{w}})$ in each of the sets $Q_{\bar{w}}^i \in \mathcal{Q}_{ln}$. Consider the ln -th dynamical partition \mathcal{P}_{ln} for the pair T_* as defined in Section 1.3. Consider a piecewise-constant map φ_l sending the element of the partition with a multi-index \bar{w} to $(x_{\bar{w}}, y_{\bar{w}})$. By (4.2), the diameters of the sets $Q_{\bar{w}}^i$ decrease at a geometric rate. Thus, the maps φ_l converge uniformly to a continuous map φ of the interval $[-1, \rho_*]$ which is a homeomorphism onto the image. Set

$$\varphi([-1, \rho_*]) \equiv \gamma.$$

By construction,

$$\varphi \circ T_* = Z \circ \varphi,$$

and the curve γ is the attractor for the pair Z . Clearly, the conjugacy φ cannot be C^1 -smooth, since the limiting pair ζ_* has a critical point at the origin. □

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